Transverse momentum distribution of hadrons in the Tsallis statistics

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Transverse momentum distributions of hadrons at high energies

Experiment:


Statistical Theory:

Tsallis-factorized distribution

\[ 1 + (q_c - 1) \frac{\varepsilon - \mu}{T} \left[ \frac{1}{1-q_c} \right]^{q_c} \]

Boltzmann-Gibbs distribution

\[ q_c \to 1 \]

J. Cleymans, D. Worku,

\[ \varepsilon = m_T \cosh y, \quad m_T = \sqrt{p_T^2 + m^2} \]

Is the Tsallis-factorized distribution related to the Tsallis statistics?
### What is the Tsallis statistics?

#### 1.) Definitions:

<table>
<thead>
<tr>
<th>Boltzmann-Gibbs Statistics</th>
<th>Tsallis-1 Statistics</th>
<th>Tsallis-2 Statistics</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S = -\sum_i p_i \ln p_i, \quad q = 1$</td>
<td>$S = -\sum_i \frac{p_i - p_i^q}{1 - q}, \quad 0 &lt; q &lt; \infty$</td>
<td>$S = -\sum_i \frac{p_i - p_i^{q_c}}{1 - q_c}, \quad 0 &lt; q_c &lt; \infty$</td>
</tr>
<tr>
<td>$\sum_i p_i = 1$</td>
<td>$\sum_i p_i = 1$</td>
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</tr>
<tr>
<td>$E = \sum_i p_i E_i$</td>
<td>$E = \sum_i p_i E_i$</td>
<td>$E = \sum_i p_i^{q_c} E_i$</td>
</tr>
<tr>
<td>$\langle N \rangle = \sum_i p_i N_i$</td>
<td>$\langle N \rangle = \sum_i p_i N_i$</td>
<td>$\langle N \rangle = \sum_i p_i^{q_c} N_i$</td>
</tr>
</tbody>
</table>

$p_i$ — probability of $i$-th microstate of the system

#### 2.) Legendre Transform:

$$
\Omega = E - TS - \mu \langle N \rangle
$$

#### 3.) Thermodynamic potentials:

- **B-G**
  $$
  \Omega = T \sum_i p_i \left[ \ln p_i + \frac{E_i - \mu N_i}{T} \right]
  $$

- **T-1**
  $$
  \Omega = T \sum_i p_i \left[ \frac{1 - p_i^{q-1}}{1 - q} + \frac{E_i - \mu N_i}{T} \right]
  $$

- **T-2**
  $$
  \Omega = T \sum_i p_i^{q_c} \left[ \frac{p_i^{1-q_c} - 1}{1 - q_c} + \frac{E_i - \mu N_i}{T} \right]
  $$
What is the Tsallis statistics?

4.) Constrained Local Extrema of the Thermodynamic Potential (Method of Lagrange Multipliers):

\[ \Phi = \Omega - \lambda \phi, \quad \phi = \sum_i p_i - 1 = 0 \]

\[ \frac{\partial \Phi}{\partial p_i} = 0 \]

- Lagrange function
- constrained equation
- extremization

5.) Many-body distribution functions (Probabilities of Microstates of the System) and the norm functions:

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<th>Tsallis-2 Statistics</th>
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<tr>
<td>[ p_i = \frac{1}{Z} \exp \left( - \frac{E_i - \mu N_i}{T} \right) ]</td>
<td>[ p_i = \frac{1}{Z} \left[ 1 + \frac{q-1}{q} \frac{\Lambda - E_i + \mu N_i}{T} \right]^{\frac{1}{q-1}} ]</td>
<td>[ p_i = \frac{1}{Z} \left[ 1 - (1 - q_c) \frac{E_i - \mu N_i}{T} \right]^{\frac{1}{1-q_c}} ]</td>
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<tr>
<td>[ Z = \sum_i \exp \left( - \frac{E_i - \mu N_i}{T} \right) ]</td>
<td>[ \sum_i \left[ 1 + \frac{q-1}{q} \frac{\Lambda - E_i + \mu N_i}{T} \right]^{\frac{1}{q-1}} = 1 ]</td>
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</tr>
<tr>
<td>[ \Omega = -T \ln Z = \lambda - T ]</td>
<td>[ \Lambda = \lambda - T ]</td>
<td>[ -Tq_c \frac{Z^{1-q_c} - 1}{1-q_c} = \lambda - T ]</td>
</tr>
</tbody>
</table>
What is the Tsallis-factorized statistics?

### Boltzmann-Gibbs Statistics

**Ideal Gas (Maxwell-Boltzmann):**

\[
\langle n_{\tilde{p}\sigma} \rangle = e^{\frac{\varepsilon_{p} - \mu}{T}}
\]

\[
S = -\sum_{\tilde{p}\sigma} \left[ \langle n_{\tilde{p}\sigma} \rangle \ln \langle n_{\tilde{p}\sigma} \rangle - \langle n_{\tilde{p}\sigma} \rangle \right]
\]

\[
\langle N \rangle = \sum_{\tilde{p}\sigma} \langle n_{\tilde{p}\sigma} \rangle
\]

\[
E = \sum_{\tilde{p}\sigma} \langle n_{\tilde{p}\sigma} \rangle \varepsilon_{\tilde{p}}
\]

\[
\Omega = E - TS - \mu \langle N \rangle
\]

\[
= T \sum_{\tilde{p}\sigma} \langle n_{\tilde{p}\sigma} \rangle \left[ \ln \langle n_{\tilde{p}\sigma} \rangle - 1 + \frac{\varepsilon_{\tilde{p}} - \mu}{T} \right]
\]

\[
\frac{\partial \Omega}{\partial \langle n_{\tilde{p}\sigma} \rangle} = 0, \quad \langle n_{\tilde{p}\sigma} \rangle = e^{\frac{\varepsilon_{p} - \mu}{T}}
\]

---

### Tsallis-factorized Statistics

**Ideal Gas (Maxwell-Boltzmann):**

\[
\]

\[
q_c \quad \text{real parameter}
\]

\[
S = -\sum_{\tilde{p}\sigma} \left[ f^{q_c}_{\tilde{p}\sigma} \ln_{q_c} f^{q_c}_{\tilde{p}\sigma} - f^{q_c}_{\tilde{p}\sigma} \right], \quad f^{q_c}_{\tilde{p}\sigma} \equiv \langle n_{\tilde{p}\sigma} \rangle
\]

\[
\langle N \rangle = \sum_{\tilde{p}\sigma} f^{q_c}_{\tilde{p}\sigma}
\]

\[
\ln_{q_c} (x) = \frac{x^{1-q_c} - 1}{1-q_c}, \quad 0 < q_c < \infty
\]

\[
E = \sum_{\tilde{p}\sigma} f^{q_c}_{\tilde{p}\sigma} \varepsilon_{\tilde{p}}
\]

\[
\Omega = E - TS - \mu \langle N \rangle
\]

\[
= T \sum_{\tilde{p}\sigma} f^{q_c}_{\tilde{p}\sigma} \left[ q_c \ln_{q_c} f^{q_c}_{\tilde{p}\sigma} - 1 + \frac{\varepsilon_{\tilde{p}} - \mu}{T} \right]
\]

\[
\frac{\partial \Omega}{\partial f^{q_c}_{\tilde{p}\sigma}} = 0, \quad \langle n_{\tilde{p}\sigma} \rangle = \left[ 1 + (q_c - 1) \frac{\varepsilon_{\tilde{p}} - \mu}{T} \right]^{\frac{q_c}{1-q_c}}
\]

---

- The constrained maximization of the entropy of the ideal gas (generalized from the Boltzmann-Gibbs entropy of the ideal gas) with respect to the single-particle distribution function should lead to the results of the Tsallis-2 statistics
- Is it indeed the Tsallis-factorized distribution equivalent to the distribution of the Tsallis-2 statistics?
- The Tsallis-factorized statistics should be equivalent to the Tsallis-2 statistics
1. **Tsallis-2 Statistics:**

   - **Exact results:**
     \[
     Z = \frac{1}{\pi^2} \sum_{N=1}^{1-q_c} \tilde{\omega}^N N! \frac{\Gamma\left(\frac{1}{q_c-1} - 3N\right)}{(q_c-1)^{3N} \Gamma\left(\frac{1}{q_c-1}\right)} \left[1 - \left(\frac{1}{q_c-1}\right) \frac{\mu N}{T}\right]^{\frac{1}{1-q_c} + 3N}
     \]
     \[
     \langle n_{\tilde{p}\sigma} \rangle = \frac{1}{Z} \left[1 + (q_c - 1) \frac{\epsilon_p - \mu}{T}\right]^{\frac{q_c}{1-q_c}} + \frac{1}{Z} \sum_{N=1}^{1-q_c} \tilde{\omega}^N N! \frac{\Gamma\left(\frac{q_c}{q_c-1} - 3N\right)}{(q_c-1)^{3N} \Gamma\left(\frac{q_c}{q_c-1}\right)} \left[1 + (q_c - 1) \frac{\epsilon_p - \mu(N+1)}{T}\right]^{\frac{q_c}{1-q_c} + 3N}
     \]

   - **Zeroth term approximation:**
     \[
     N = 0, \quad Z = 1
     \]
     \[
     \langle n_{\tilde{p}\sigma} \rangle = \left[1 + (q_c - 1) \frac{\epsilon_p - \mu}{T}\right]^{\frac{q_c}{1-q_c}}
     \]

2. **Tsallis-factorized Statistics:**

   - The constrained maximization of the Tsallis-factorized entropy of the ideal gas (generalized from the Boltzmann-Gibbs entropy of the ideal gas) with respect to the single-particle distribution function does not lead to the true results for the Tsallis-2 statistics.

   - **Exact results:**
     \[
     \langle n_{\tilde{p}\sigma} \rangle = \left[1 + (q_c - 1) \frac{\epsilon_p - \mu}{T}\right]^{\frac{q_c}{1-q_c}}
     \]

   - **Zeroth term approximation:**
     \[
     \langle n_{\tilde{p}\sigma} \rangle = \left[1 + (q_c - 1) \frac{\epsilon_p - \mu}{T}\right]^{\frac{q_c}{1-q_c}}
     \]

- The partition function is divergent
- We truncate the series
- In the partition function and the mean occupation numbers only the physical terms are preserved
- The mean occupation numbers in the Tsallis-2 statistics
- The zeroth term approximation is valid only for \( q_c > 3/2 \)
- The Tsallis-factorized distribution is not equivalent to the distribution of the Tsallis-2 statistics
- The Tsallis-factorized statistics is not equivalent to the Tsallis-2 statistics
- The Tsallis-factorized statistics can serve as a particular statistics independent from the Tsallis statistics

**References:**

Ultrarelativistic Ideal Gas: Tsallis-1 statistics $q < 1$

- **The norm equation:**
  - The norm equation is divergent
  
  $\sum_{N=0}^{N_0} \phi(N) + \sum_{N=N_0+1}^{\infty} \phi(N) = 1$

  $\phi(N) = \frac{\tilde{\omega}^N}{N!} \left( \frac{q}{1-q} \right)^{3N} \frac{\Gamma \left( \frac{1}{1-q} - 3N \right)}{\Gamma \left( \frac{1}{1-q} \right)} \left[ 1 + \frac{q-1}{q} \frac{\Lambda + \mu N}{T} \right]^{\frac{1}{q-1}+3N}$

  $\tilde{\omega} = \frac{gVT^3}{\pi^2}$

  $T = 100MeV$, $R = 4 fm$, $\mu = 0$

- **Regularization:**
  - We truncate the series
  - In the norm equation only the physical terms are preserved

  $\sum_{N=0}^{N_0} \phi(N) + \sum_{N=N_0+1}^{\infty} \phi(N) = 1$

- **The cut-off prescription:**
  - The inflection point

  $\frac{\partial^2 \ln \phi(N)}{\partial N^2} \bigg|_{N=N_0} = 0$

  $N_0$ — the upper bound of summation

- **Table:**

<table>
<thead>
<tr>
<th>$q$</th>
<th>0.9</th>
<th>0.95</th>
<th>0.99</th>
<th>0.995</th>
<th>0.999</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_0$</td>
<td>0</td>
<td>1</td>
<td>7</td>
<td>16</td>
<td>82</td>
</tr>
</tbody>
</table>
Ultrarelativistic Ideal Gas: Tsallis-1 statistics

1. Tsallis-1 Statistics:

**Exact results:**

\[
\langle n_{\tilde{p}\sigma} \rangle = \left[ 1 + \frac{q-1}{q} \frac{\Lambda - \varepsilon_{\tilde{p}} + \mu}{T} \right]^{\frac{1}{q-1}} + \sum_{N=1}^{N_0} \tilde{\omega}^N \frac{\Gamma \left( \frac{1}{1-q} \right)}{N!} \left[ 1 + \frac{q-1}{q} \frac{\Lambda + \mu N}{T} \right]^{\frac{1}{q-1}+3N} = 1
\]

- The mean occupation numbers in the Tsallis-1 statistics

**Zeroth term approximation:** (Definition: All terms with \( N \geq 1 \) in the series given above are deleted by hand)

\[
N = 0, \quad \Lambda = 0
\]

\[
\langle n_{\tilde{p}\sigma} \rangle = \left[ 1 + \frac{q-1}{q} \frac{\varepsilon_{\tilde{p}} - \mu}{T} \right]^{\frac{1}{q-1}}
\]

2. Tsallis-factorized Statistics:

- The Tsallis-factorized distribution is not equivalent to the distribution of the Tsallis-1 statistics

The mean occupation numbers of the Tsallis-factorized statistics

\[
\langle n_{\tilde{p}\sigma} \rangle = \left[ 1 + (q_c - 1) \frac{\varepsilon_{\tilde{p}} - \mu}{T} \right]^{\frac{q_c}{1-q_c}}
\]

- The Tsallis-factorized statistics is not equivalent to the Tsallis statistics (Tsallis-1 and Tsallis-2 statistics)

\[q \to 1 / q_c\]

Comparison of Tsallis-factorized statistics with Tsallis-1 statistics: Charged pions

\( p + p \)

A.S.P., arXiv:1608.01888


Ultrarelativistic distributions of the Tsallis-1 statistics:

\[
\frac{d^2 N}{dp_T\,dy} \bigg|_{\gamma_0} = \frac{gV}{(2\pi)^2} p_T^2 \int_{\gamma_0} dy \cosh y \sum_{N=0}^{N_0} \tilde{\omega}^N \frac{q}{1-q} \frac{\Gamma \left( \frac{1}{1-q} - 3N \right)}{N!} \frac{1}{\Gamma \left( \frac{1}{1-q} \right)} \left[ 1 + \frac{q-1}{q} \Lambda - p_T \cosh y + \mu(N+1) \right]^{\frac{1}{q-1} + 3N} \]

\[
\frac{1}{2\pi p_T} \frac{d^2 N}{dp_T\,dy} = -\frac{gV}{(2\pi)^3} p_T \cosh y \sum_{N=0}^{N_0} \tilde{\omega}^N \frac{q}{1-q} \frac{\Gamma \left( \frac{1}{1-q} - 3N \right)}{N!} \frac{1}{\Gamma \left( \frac{1}{1-q} \right)} \left[ 1 + \frac{q-1}{q} \Lambda - p_T \cosh y + \mu(N+1) \right]^{\frac{1}{q-1} + 3N} \]
Comparison of Tsallis-factorized statistics with Tsallis-1 statistics: Charged pions

\[ p + p \]


Ultrarelativistic distributions of the Tsallis-1 statistics:

\[
\frac{d^2N}{d\ln p_T dy} \bigg|_{y_0} = \frac{gV}{(2\pi)^2} p_T^2 \int_{y_0}^{y_1} dy \cosh y \left( \sum_{N=0}^{N_N} (\frac{q}{1-q})^N \right) \frac{\Gamma \left( \frac{1}{1-q} - 3N \right)}{\Gamma \left( \frac{1}{1-q} \right)} 
\]

\[
\frac{1}{2\pi p_T} \frac{d^2N}{dp_T dy} \bigg|_{y_0} = \frac{gV}{(2\pi)^2} p_T \int_{y_0}^{y_1} dy \cosh y \left( \sum_{N=0}^{N_N} (\frac{q}{1-q})^N \right) \frac{\Gamma \left( \frac{1}{1-q} - 3N \right)}{\Gamma \left( \frac{1}{1-q} \right)} 
\]

\[
\left[ 1 + \frac{q-1}{q} \Lambda - p_T \cosh y + \mu(N+1) \right]^{\frac{1}{q-1} + 3N} 
\]

A.S.P., arXiv:1608.01888
Energy dependence of the parameters of the Tsallis-1 statistics and the Tsallis-factorized statistics

\( p + p \)

\[ \frac{d^2N}{dp_T dy} = \frac{gVp_T^2 \cosh y}{(2\pi)^2} \left[ 1 + (q_c - 1) \frac{p_T \cosh y - \mu}{T} \right]^{q_c} \]

Solid points – Tsallis-1 statistics
Open symbols – Tsallis-factorized statistics

Tsallis-factorized distribution
(the zeroth term approximation, \( q_c = \frac{1}{q} \))


A.S.P., arXiv:1608.01888
Parameters of the Tsallis-1 statistics fit for the pions produced in $pp$ collisions at different energies

<table>
<thead>
<tr>
<th>Collaboration</th>
<th>Type</th>
<th>$\sqrt{s}$, GeV</th>
<th>$T$, MeV</th>
<th>$R$, fm</th>
<th>$q$</th>
<th>$\chi^2/ndf$</th>
</tr>
</thead>
<tbody>
<tr>
<td>NA61/SHINE</td>
<td>$\pi^-$</td>
<td>6.3</td>
<td>85.78±10.79</td>
<td>4.047±0.235</td>
<td>0.9623±0.0142</td>
<td>2.821/15</td>
</tr>
<tr>
<td>NA61/SHINE</td>
<td>$\pi^-$</td>
<td>7.7</td>
<td>79.05±8.01</td>
<td>4.304±0.204</td>
<td>0.9505±0.0107</td>
<td>1.472/15</td>
</tr>
<tr>
<td>NA61/SHINE</td>
<td>$\pi^-$</td>
<td>8.8</td>
<td>82.01±9.28</td>
<td>4.294±0.212</td>
<td>0.9542±0.0123</td>
<td>1.821/15</td>
</tr>
<tr>
<td>NA61/SHINE</td>
<td>$\pi^-$</td>
<td>12.3</td>
<td>75.47±7.41</td>
<td>4.627±0.253</td>
<td>0.9451±0.0083</td>
<td>1.152/15</td>
</tr>
<tr>
<td>NA61/SHINE</td>
<td>$\pi^-$</td>
<td>17.3</td>
<td>95.83±6.38</td>
<td>4.798±0.246</td>
<td>0.9326±0.0166</td>
<td>0.865/15</td>
</tr>
<tr>
<td>PHENIX</td>
<td>$\pi^+$</td>
<td>62.4</td>
<td>97.62±11.92</td>
<td>3.744±0.648</td>
<td>0.9197±0.0093</td>
<td>1.654/23</td>
</tr>
<tr>
<td>PHENIX</td>
<td>$\pi^-$</td>
<td>62.4</td>
<td>93.76±11.69</td>
<td>3.971±0.716</td>
<td>0.9184±0.0091</td>
<td>0.878/23</td>
</tr>
<tr>
<td>PHENIX</td>
<td>$\pi^+$</td>
<td>200.0</td>
<td>79.89±11.80</td>
<td>4.247±0.899</td>
<td>0.8894±0.0082</td>
<td>0.987/24</td>
</tr>
<tr>
<td>PHENIX</td>
<td>$\pi^-$</td>
<td>200.0</td>
<td>87.20±11.48</td>
<td>3.823±0.714</td>
<td>0.8965±0.0081</td>
<td>0.691/24</td>
</tr>
<tr>
<td>ALICE</td>
<td>$\pi^+$</td>
<td>900.0</td>
<td>82.72±2.01</td>
<td>3.965±0.069</td>
<td>0.8766±0.0037</td>
<td>3.609/30</td>
</tr>
<tr>
<td>ALICE</td>
<td>$\pi^-$</td>
<td>900.0</td>
<td>83.92±2.02</td>
<td>3.918±0.068</td>
<td>0.8790±0.0036</td>
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<tr>
<td>ALICE</td>
<td>$\pi^+ + \pi^-$</td>
<td>2760.0</td>
<td>90.61±1.45</td>
<td>3.496±0.057</td>
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<tr>
<td>ALICE</td>
<td>$\pi^+ + \pi^-$</td>
<td>7000.0</td>
<td>78.75±1.86</td>
<td>4.606±0.093</td>
<td>0.8533±0.0024</td>
<td>9.775/38</td>
</tr>
</tbody>
</table>

A.S.P., arXiv:1608.01888
Parameters of the Tsallis-factorized statistics

\( p + p \)

A.S.P., arXiv:1608.01888

Parameters of the fit by the distribution of the Tsallis-factorized statistics (the zero term approximation of Tsallis-1 statistics) for the pions produced in pp collisions at different energies

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<td>NA61/SHINE</td>
<td>( \pi^- )</td>
<td>6.3</td>
<td>99.59±7.32</td>
<td>4.045±0.234</td>
<td>0.9563 0.0190</td>
<td>1.0457±0.0208</td>
<td>2.825/15</td>
</tr>
<tr>
<td>NA61/SHINE</td>
<td>( \pi^- )</td>
<td>7.7</td>
<td>96.93±6.49</td>
<td>4.300±0.222</td>
<td>0.9400 0.0171</td>
<td>1.0638±0.0194</td>
<td>1.481/15</td>
</tr>
<tr>
<td>NA61/SHINE</td>
<td>( \pi^- )</td>
<td>8.8</td>
<td>99.37±6.29</td>
<td>4.290±0.204</td>
<td>0.9455 0.0172</td>
<td>1.0576±0.0193</td>
<td>1.838/15</td>
</tr>
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<td>NA61/SHINE</td>
<td>( \pi^- )</td>
<td>12.3</td>
<td>95.92±6.29</td>
<td>4.619±0.228</td>
<td>0.9324 0.0170</td>
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<td>1.175/15</td>
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<td>900.0</td>
<td>82.72±2.01</td>
<td>3.965±0.069</td>
<td>0.8766 0.0037</td>
<td>1.1408±0.0048</td>
<td>3.609/30</td>
</tr>
<tr>
<td>ALICE</td>
<td>( \pi^- )</td>
<td>900.0</td>
<td>83.92±2.02</td>
<td>3.918±0.068</td>
<td>0.8790 0.0036</td>
<td>1.1376±0.0047</td>
<td>1.610/30</td>
</tr>
<tr>
<td>ALICE</td>
<td>( \pi^+ + \pi^- )</td>
<td>2760.0</td>
<td>90.61±1.45</td>
<td>3.496±0.057</td>
<td>0.8726 0.0012</td>
<td>1.1460±0.0016</td>
<td>12.18/60</td>
</tr>
<tr>
<td>ALICE</td>
<td>( \pi^+ + \pi^- )</td>
<td>7000.0</td>
<td>78.75±1.86</td>
<td>4.606±0.093</td>
<td>0.8533 0.0024</td>
<td>1.1719±0.0032</td>
<td>9.775/38</td>
</tr>
</tbody>
</table>

- The results of the Tsallis-factorized statistics (the zeroth term approximation of the Tsallis-1 statistics) deviate from the results of the Tsallis-1 statistics only at low NA61/SHINE energies when the value of the parameter \( q \) is close to unity.
- At higher energies, when the value of the parameter \( q \) deviates essentially from the unity, the Tsallis-factorized statistics satisfactorily recovers the results of the Tsallis-1 statistics because at this values of \( q \) in the series of the Tsallis-1 statistics only one term \( N = 0 \) is physical.
Conclusions

1. The analytical expressions for the ultrarelativistic transverse momentum distribution of the Tsallis-1 and Tsallis-2 statistics were obtained.

2. We found that the transverse momentum distribution of the Tsallis-factorized statistics in the ultrarelativistic case is not equivalent to the transverse momentum distribution of both the Tsallis-1 and Tsallis-2 statistics.

3. The transverse momentum distribution of the Tsallis-factorized statistics is equivalent only to the distribution in the zeroth term approximation of the Tsallis-2 statistics and the Tsallis-1 statistics with transformation of the parameter $q$ to $1/q$.

4. We have demonstrated on the base of the ultrarelativistic ideal gas that the Tsallis–factorized statistics is not equivalent to the Tsallis statistics (Tsallis-1 and Tsallis-2 statistics).

5. The Tsallis-factorized statistics is a particular statistics independent from the Tsallis statistics (Tsallis-1 and Tsallis-2 statistics).
Thank you for your attention