Higher-order perturbative coefficients in QCD from series acceleration by conformal mappings

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Outline

1. Low-order calculations in perturbative QCD
2. Large-order behaviour
3. Hyperasymptotics, transseries, resurgence
4. Series acceleration by conformal mappings
5. Prediction of higher-order perturbative coefficients
6. Summary and conclusions

Hadronic vacuum polarization

- Electromagnetic current $J^\mu(x)$ of light hadrons (π and K mesons)

- Lorentz-invariant vacuum polarization amplitude $\Pi(s)$:

$$-i \int \mathrm{d}^4x \, e^{iq\cdot x} \langle 0| \, T\, \{J^\mu(x), J^\nu(0)^\dagger\} \, |0\rangle = (q^\mu q^\nu - g^{\mu\nu} q^2) \, \Pi(s), \quad s = q^2$$

- Causality and unitarity: for $s \geq 4m_\pi^2$, $\Pi(s)$ is complex and

$$\mathrm{Im}\, \Pi(s) \approx \sigma(e^+ e^- \to \text{hadrons}), \quad \mathrm{Im}\, \Pi(s) \approx \sigma(\tau \to \nu_\tau \text{hadrons})$$

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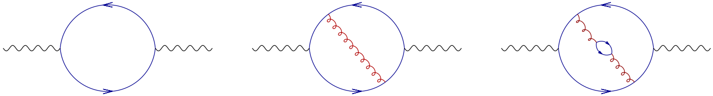
- Perturbative QCD: Feynman graphs with free quark and gluon lines



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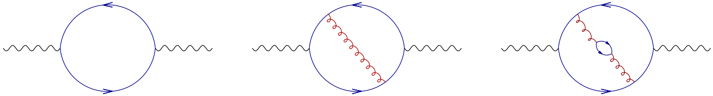

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- State of the art: calculations in perturbative QCD up to five-loop order

$$\Pi(s) \sim \alpha_s^4, \qquad \alpha_s = \frac{g^2}{4\pi}$$

Perturbative QCD

- Renormalization-group invariant quantity (Adler function)

$$D(s) = -s\,\frac{d\Pi(s)}{ds}, \qquad \widehat{D}(s) \equiv 4\pi^2 D(s) - 1.$$

- Standard expansion in powers of the renormalized coupling $\alpha_s(\mu^2)$:

$$\widehat{D}(s) = \sum_{n\geq 1} (\alpha_s(\mu^2)/\pi)^n \sum_{k=1}^{n} k\, c_{n,k}\, L^{k-1}, \qquad L = \ln(-s/\mu^2)$$

- Renormalization-group improved expansion: $\mu^2 = -s > 0 \quad \Rightarrow$

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  - Coefficients calculated in $\overline{\text{MS}}$ renormalization scheme:

$$\boxed{c_{1,1} = 1, \quad c_{2,1} = 1.640, \quad c_{3,1} = 6.371, \quad c_{4,1} = 49.076}$$

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  - Estimates of next coefficients of interest for testing the Standard Model at intermediate energies   motivation of the present work!

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- Large-order behaviour: $\boxed{c_{n,1} \sim n!}$ for $n \to \infty$

Problems of perturbative QCD

- The description of physical hadronic observables is not straightforward
  - The expansions truncated at finite orders depend on the renormalization scheme and scale
  - Perturbative QCD is valid in the Euclidian region $s < 0$, far from the hadronic thresholds
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- Additional terms might be necessary for recovering the exact function

- Consider an asymptotic expansion for $z \to 0_+$ to a continuous function $F(z)$:

$$F(z) \simeq a_0 + a_1 z + a_2 z^2 + \ldots \qquad |F(z) - \sum_0^N a_n z^n| = O(z^{N+1}), \;\; N = 1, 2, \ldots \;\; z \to 0_+$$

- Remark: for an arbitrary $c > 0$

$$e^{-c/z} \simeq 0 + 0 \times z + 0 \times z^2 + \ldots \qquad (z > 0)$$

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- Hyperasymptotics
  - expand a well behaved function as an asymptotic (divergent) series
  - add terms exponentially-small in the coupling of the original series
  - add terms exponentially-small in the coupling of the second series
  - continue the process ("transseries")
  - this will allow the expanded function to "resurge"

- The dependence of the coupling on the scale μ^2 given by RGE:

$$-\mu^2 \frac{d\alpha_s}{d\mu^2} \equiv \beta(\alpha_s) = \beta_0\, \alpha_s^2 + \beta_1\, \alpha_s^3 + \beta_2\, \alpha_s^4 + \beta_3\, \alpha_s^5 + \ldots$$

- The running coupling to one-loop: $\boxed{s = -Q^2}$

$$\boxed{\alpha_s(Q^2) \approx \frac{1}{\beta_0 \ln(Q^2/\Lambda^2)}}$$

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$\Longrightarrow$ Hyperasymptotic perturbative expansion in QCD:

$$\widehat{D}(s) \simeq \sum_{n\geq 1} c_{n,1}\, (\alpha_s(Q^2)/\pi)^n \quad + \quad \sum_{k\geq 1} \frac{C_k}{Q^{2k}} \quad + \quad \sum_{j\geq 1} D_j e^{-F_j Q^2}$$

pure PT "power corrections" "duality violating" terms

Laplace-Borel transform

- Starting from a factorially divergent series, define a convergent series:

$$\widehat{D} = \sum_{n\geq 1} c_{n,1}(\alpha_s/\pi)^n \quad \Rightarrow \quad B_D(u) = \sum_{n=0}^{\infty} b_n\, u^n, \qquad b_n = \frac{c_{n+1,1}}{\beta_0^n\, n!}$$

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- The large-order behaviour encoded in the singularities of $B_D(u)$
  - branch-points on the real semiaxis $u \geq 2$ (infrared renormalons)
  - branch-points on the real semiaxis $u \leq -1$ (ultraviolet renormalons)
- The nature of the first branch points at $u = -1$ and $u = 2$ is known:

$$B_D(u) = O((1+u)^{-\gamma_1}), \qquad \gamma_1 = 1.21$$
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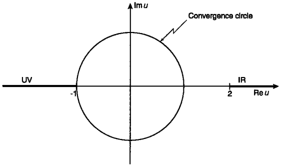
- Convergence region in the u plane:



Recover the original function by the Laplace-Borel integral

$$\widehat{D}(s) = \frac{1}{\beta_0} \int_0^\infty \exp\left(\frac{-u}{\beta_0 \alpha_s(-s)}\right) B_D(u)\, du$$

- The integral is not defined due to the singularities of $B_D(u)$ for $u \geq 2$

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- The standard expansions fail to deal with the singularities in the Borel plane

$\Longrightarrow$ Consider alternative expansions which implement these singularities

Series acceleration by conformal mappings

- Series acceleration: increase the convergence domain and the convergence rate of an expansion
- A power series convergent in a disk of positive radius around the origin, is replaced by a series in powers of another variable, which performs the conformal mapping of the original complex plane (or a part of it) onto a disk of radius equal to unity

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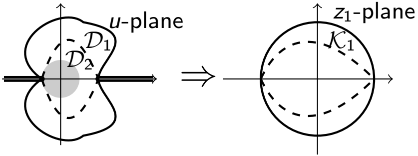
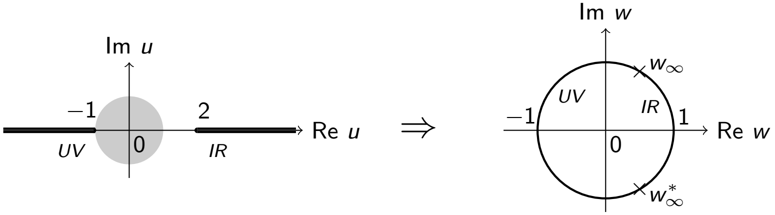
Larger domain mapped onto the unit disk ⇒ higher convergence rate



$$\left| \frac{a_{n,1}\,(\tilde{z}_1(u))^n}{a_{n,2}\,(\tilde{z}_2(u)^n} \right| < 1$$

Optimal conformal mapping $\tilde{w}(u)$: whole holomorphy domain ⇒ $|w| < 1$

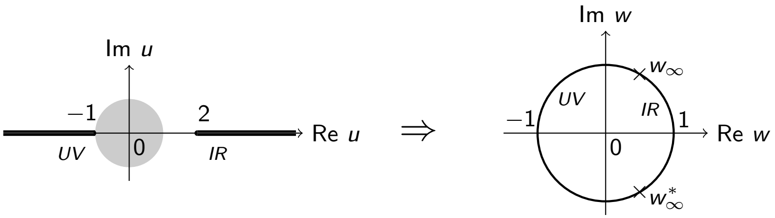
Optimal conformal mapping of Borel plane



Achieved by $w \equiv \tilde{w}(u)$, $\tilde{w}(0) = 0$, and the inverse $\tilde{u}(w)$:

$$\boxed{\tilde{w}(u) = \frac{\sqrt{1+u} - \sqrt{1-u/2}}{\sqrt{1+u} + \sqrt{1-u/2}}} \qquad \boxed{\tilde{u}(w) = \frac{8w}{3-2w+3w^2}}$$

(IC & Fischer, 1999)



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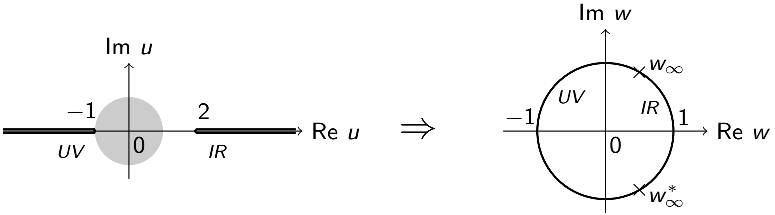

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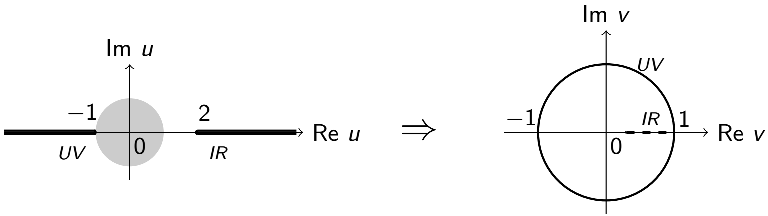
- Optimal expansion with singularity softening (s.s.):

$$B_D(u) = \frac{1}{(1+w)^{2\gamma_1}(1-w)^{2\gamma_2}} \sum_{n\geq 0} \bar{c}_n\, w^n$$

Mapping which accounts only for the UV reormalons



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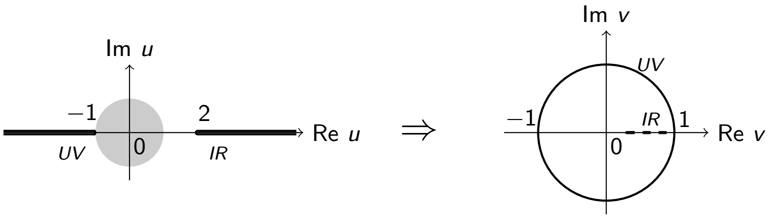

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- Alternative expansion of the Borel transform:

$$B_D(u) = \sum_{n\geq 0} f_n \, v^n,$$

- Expansion with singularity softening:

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Prediction of higher-order coefficients

- Four known coefficients $c_{n,1}$, $1 \leq n \leq 4$ $\Rightarrow$ four coefficients b_n, $0 \leq n \leq 3$
- Can one predict higher-order coefficients?
- Use theoretical knowledge on the expanded function
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- insert $u = \tilde{u}(w)$ in this truncated expansion
- expand its product with the global prefactor $(1+w)^{2\gamma_1}(1-w)^{2\gamma_2}$ in powers of w to the same order $N-1$
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$\Longrightarrow$

- recover the first N input coefficients
- obtain also definite values for the higher-order coefficients

Coefficient $c_{N,1}$ from input $c_{n,1}$, $n \leq N-1$ for a mathematical model

N	Series in v^n	Series in w^n	v^n and s.s.	w^n and s.s.	Exact $c_{N,1}$
4	-52.34	-17.61	14.77	17.85	49.076
5	-932.45	-270.46	255.98	255.73	$283.$
6	-14348.46	-2290.94	3096.35	2928.76	3275.45
7	$-274384.$	-39054.7	15740.1	16308.73	18758.4
8	-5.12×10^6	-272605.1	350336.4	381151.6	388445.6
9	-1.14×10^8	-6.89×10^6	455072.1	963059.1	919119.2
10	-2.56×10^9	-1.424×10^7	7.82×10^7	8.49×10^7	8.37×10^7
11	-6.68×10^{10}	-1.78×10^9	-5.74×10^8	-5.04×10^8	-5.19×10^8
12	-1.76×10^{12}	1.66×10^{10}	3.36×10^{10}	3.39×10^{10}	3.38×10^{10}
13	-5.29×10^{13}	-8.47×10^{11}	-5.89×10^{11}	-6.04×10^{11}	-6.04×10^{11}
14	-1.61×10^{15}	1.98×10^{13}	2.42×10^{13}	2.34×10^{13}	2.34×10^{13}
15	-5.48×10^{16}	-7.09×10^{14}	-6.24×10^{14}	-6.53×10^{14}	-6.52×10^{14}
16	-1.89×10^{18}	2.32×10^{16}	2.52×10^{16}	2.42×10^{16}	2.42×10^{16}
17	-7.22×10^{19}	-8.62×10^{17}	-8.12×10^{17}	-8.46×10^{17}	-8.46×10^{17}
18	-2.78×10^{21}	3.33×10^{19}	3.48×10^{19}	3.36×10^{19}	3.36×10^{19}
19	-1.18×10^{23}	-1.36×10^{21}	-1.32×10^{21}	-1.36×10^{21}	-1.36×10^{21}
20	-5.01×10^{24}	5.90×10^{22}	6.07×10^{22}	5.92×10^{22}	5.92×10^{22}
21	-2.34×10^{26}	-2.68×10^{24}	-2.62×10^{24}	-2.68×10^{24}	-2.68×10^{24}
22	-1.09×10^{28}	1.28×10^{26}	1.31×10^{26}	1.28×10^{26}	1.28×10^{26}
23	-5.54×10^{29}	-6.41×10^{27}	-6.32×10^{27}	-6.41×10^{27}	-6.41×10^{27}
24	-2.80×10^{31}	3.35×10^{29}	3.39×10^{29}	3.35×10^{29}	3.35×10^{29}
25	-1.54×10^{33}	-1.83×10^{31}	-1.81×10^{31}	-1.83×10^{31}	-1.83×10^{31}

Higher-order coefficients from $B_D(u)$

- Standard truncated expansion with 4 given coefficients:

$$B_D(u) = 1 + 0.7288\,u + 0.6292\,u^2 + 0.7181\,u^3$$

- The above algorithm leads to the optimal expansion

$$B_D(u) = \frac{1 - 0.7973\,w + 0.4095\,w^2 + 8.6647\,w^3}{(1+w)^{2\gamma_1}(1-w)^{2\gamma_2}}$$

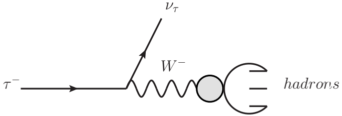
- Reexpanded in powers of u, it gives

$$\begin{aligned} B_D(u) &= 1 + 0.7288\,u + 0.6292\,u^2 + 0.7181\,u^3 \\ &+ 0.4157\,u^4 + 0.4220\,u^5 + 0.1429\,u^6 + \dots \end{aligned}$$

  - The first four coefficients reproduce the input values
  - The remaining coefficients lead to:

$$c_{5,1} = 255.73, \quad c_{6,1} = 2920.2, \quad c_{7,1} = 13357.1\,.$$

- Hadronic decay of the τ lepton:



$$R_\tau = \frac{\Gamma(\tau^- \to \text{hadrons } \nu_\tau)}{\Gamma(\tau^- \to e\bar{\nu}_e\nu_\tau)} = C_{EW}\,(1 + \delta^{(0)})$$

$\delta^{(0)}$: hadronic contribution

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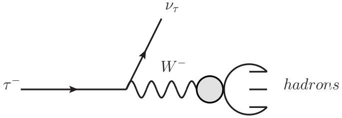
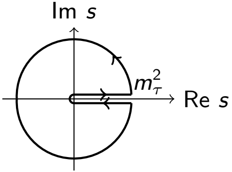


$$R_\tau = \frac{\Gamma(\tau^- \to \text{hadrons}\ \nu_\tau)}{\Gamma(\tau^- \to e\bar{\nu}_e\nu_\tau)} = C_{EW}\,(1 + \delta^{(0)})$$

$\delta^{(0)}$: hadronic contribution

- Unitarity and analyticity for the hadronic polarization function $\Rightarrow$

$$\delta^{(0)} = \frac{1}{2\pi i} \oint\limits_{|s|=m_\tau^2} \frac{ds}{s} \left(1 - \frac{s}{m_\tau^2}\right)^3 \left(1 + \frac{s}{m_\tau^2}\right) \widehat{D}(s), \qquad m_\tau = 1.78\,\mathrm{GeV}$$



- Inserting in the integral the perturbative expansion of $\widehat{D}(s)$ leads to

$$\delta^{(0)} = \sum_{n\geq 1} d_n \left(\alpha_s(m_\tau^2)\right)^n$$

$d_1 = 1,\ d_2 = 5.20,\ d_3 = 26.37,\ d_4 = 127.08,\ d_5 = 307.8 + c_{5,1},$
$d_6 = -5848.2 + 17.81 c_{5,1} + c_{6,1}, d_7 = -97769.1 + 61.33\, c_{5,1} + 21.38\, c_{6,1} + c_{7,1}$

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- Borel transform of $\delta^{(0)}$:

$$B_\delta(u) = \sum_{n\geq 0} \frac{d_{n+1}}{\beta_0^n n!}\, u^n$$

Properties of $B_\delta(u)$

In the one-loop approximation for the coupling the following relation is valid:

$$B_\delta(u) = \frac{12}{(1-u)(3-u)(4-u)} \frac{\sin(\pi u)}{\pi u} B_D(u)$$

$$\boxed{B_\delta(u) \sim (1+u)(2-u)\, B_D(u)}$$

$\Rightarrow$ The nature of the singularities at $u = 2$ and $u = -1$ is modified

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But:

- Beyond the one-loop approximation the simple zeros become branch-points
- The behaviour of $B_\delta(u)$ near the first singularities is not exactly known

$\Rightarrow$ $B_\delta(u)$ is not suitable for a precise extraction of higher-order coefficients

Other contour integrals

Consider a general weighted contour integral:

$$I_\omega = \frac{1}{2\pi i} \oint\limits_{|s|=m_\tau^2} \frac{ds}{s}\, \omega(s)\, \widehat{D}(s),$$

In the one-loop approximation of the coupling:

$$B_{I_\omega}(u) = F_\omega(u)\, B_D(u)$$

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Requirements on the weight $\omega(s)$:

- $\omega(s)$ should vanish at the timelike point $s = m_\tau^2$, in order to suppress the region where the perturbative logarithms $\ln(-s/m_\tau^2)$ are large
- $F_\omega(u)$ should not vanish at $u = 2$ and $u = -1$
- $F_\omega(u)$ should not have poles or zeros at low values of $|u|$

$\Rightarrow$ $B_{I_\omega}(u)$ has the same dominant singularities as $B_D(u)$

Borel transforms

i	$\omega_i(s)$	$F_{\omega_i}(u)$
1	$\left(1-\frac{s}{m_\tau^2}\right)$	$\frac{1}{(1-u)}\frac{\sin(\pi u)}{\pi u}$
2	$\left(1-\frac{s}{m_\tau^2}\right)^2$	$\frac{2}{(1-u)(2-u)}\frac{\sin(\pi u)}{\pi u}$
3	$\left(1-\frac{s}{m_\tau^2}\right)^2\left(2+\frac{s}{m_\tau^2}\right)$	$\frac{6}{(1-u)(3-u)}\frac{\sin(\pi u)}{\pi u}$
4	$\left(1-\frac{s}{m_\tau^2}\right)^3$	$-\frac{6}{(1-u)(2-u)(3-u)}\frac{\sin(\pi u)}{\pi u}$
5	$\left(1-\frac{s}{m_\tau^2}\right)^3\left(1+\frac{s}{m_\tau^2}\right)$	$\frac{12}{(1-u)(3-u)(4-u)}\frac{\sin(\pi u)}{\pi u}$
6	$\left(1-\frac{s}{m_\tau^2}\right)^3\left(3+\frac{s}{m_\tau^2}\right)$	$\frac{24}{(1-u)(2-u)(4-u)}\frac{\sin(\pi u)}{\pi u}$
7	$\left(1-\frac{s}{m_\tau^2}\right)\frac{m_\tau^2}{s}$	$-\frac{1}{(1+u)}\frac{\sin(\pi u)}{\pi u}$
8	$\left(1-\frac{s}{m_\tau^2}\right)^2\frac{m_\tau^2}{s}$	$-\frac{2}{(1-u)(1+u)}\frac{\sin(\pi u)}{\pi u}$
9	$\left(1-\frac{s}{m_\tau^2}\right)^3\frac{m_\tau^2}{s}$	$-\frac{6}{(1-u)(2-u)(1+u)}\frac{\sin(\pi u)}{\pi u}$
10	$\left(1-\frac{s}{m_\tau^2}\right)^3\left(1+\frac{s}{m_\tau^2}\right)\frac{m_\tau^2}{s}$	$-\frac{12}{(2-u)(3-u)(1+u)}\frac{\sin(\pi u)}{\pi u}$

Higher-order coefficients from a suitable contour integral

Consider the contour integral:

$$I = \frac{1}{2\pi i} \oint_{|s|=m_\tau^2} \frac{ds}{3s} \left(\frac{s}{m_\tau^2} - 1\right)^3 \frac{m_\tau^2}{s} \widehat{D}(s)$$

Perturbative expansion:

$$I = \sum_{n\geq 1} I_n \left(\alpha_s(m_\tau^2)\right)^n \quad \Rightarrow \quad B_I(u) = \sum_{n\geq 0} \frac{I_{n+1}}{\beta_0^n n!} u^n,$$

$I_1 = 1,\ I_2 = 2.76,\ I_3 = 8.06,\ I_4 = -17.85 + c_{4,1},\ I_5 = -379.33 + 4.5\, c_{4,1} + c_{5,1},$
$I_6 = -2190.8 - 31.99\, c_{4,1} + 5.63\, c_{5,1} + c_{6,1},\ I_7 = -895.7 - 406.2\, c_{4,1} - 49.98\, c_{5,1} + 6.75\, c_{6,1} + c_{7,1}.$

- Optimal representation:

$$B_I(u) = \frac{1 - 0.536\, w - 1.168\, w^2 - 1.181\, w^3}{(1+w)^{2\gamma_1}(1-w)^{2\gamma_2}},$$

- Reexpanded in powers of u leads to the higher-order coefficients:

$$c_{5,1} = 327.0, \quad c_{6,1} = 2840.6, \quad c_{7,1} = 26475$$

Final results

- Average of the predictions obtained from the expansions of $B_D(u)$ and $B_I(u)$ in powers of w and v

- With only three input coefficients $c_{n,1}$:

$$\boxed{c_{4,1} = 34.4 \pm 19.6} \qquad \text{consistent with the true value } c_{4,1} = 49.076$$

- With four input coefficients $c_{n,1}$:

$$\boxed{c_{5,1} = 287 \pm 40, \quad c_{6,1} = 2948 \pm 208, \quad c_{7,1} = (1.89 \pm 0.75) \times 10^4}$$

- Conservative definition of the error such as to cover the range of individual values (not a statistical error)

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- Comparison with predictions based on other methods:
  - Fastest Apparent Convergence (FAC) or Principle of Minimum Sensitivity (PMS): $c_{5,1} \approx 275$
  - Qualitative trend in the expansion of the τ hadronic width: $c_{5,1} = 283 \pm 142$
  - Rational approximants of the τ hadronic width in the coupling and the Borel planes: $c_{5,1} = 277 \pm 51$, $c_{6,1} = 3460 \pm 690$, $c_{7,1} = (2.02 \pm 0.72) \times 10^4$

Summary and conclusions

- Impressive progress in perturbative QCD: calculations available to five-loop order for several observables
- However,
  - higher-order calculations not expected in the near future
  - estimates of higher-order coefficients of interest for precision tests of the Standard Model at intermediate energies
  - complications due to hyperasymptotics (especially quark-hadron duality violation) still under debate
- The series acceleration by conformal mappings of the Borel plane is a possible alternative to transseries in perturbative QCD
- The method allows reasonable predictions of the perturbative coefficients of the QCD Adler function up to eight-loop order

Hyperasymptotics, quark-hadron duality violation

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